Fourier Coefficients of Beurling Functions and a Class of Mellin Transform Formally Determined by its Values on the Even Integers

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Abstract. It is a well-known fact that Riemann Hypothesis will follows if the function identically equal to -1 can be arbitrarily approximated in the norm $\|.\|$ of $L^2([0,1],dx)$ by functions of the form $f(x) = \sum_{k=1}^{n} a_k\,\rho\left(\frac{\theta_k}{x}\right)$, where $\rho(x) := x - [x]$, and $a_k \in \mathbb{C}$, $0 < \theta_k \leqslant 1$ satisfies $\sum_{k=1}^{n} a_k\,\theta_k = 0$. Parsevall Identity $\|f(x)+1\|^2 = \sum_{n\in\mathbb{Z}} |c(n)|^2$ is a possible tool to compute or estimate this norm. In this note we give an expression for the Fourier coefficients $c(n)$ of $f+1$, when f is a function defined as above. As an application, we derive an expression for $M_f(s) := \int_0^1 (f(x)+1)\,x^{s-1}\,dx$ as a series that only depends on $M_f(2k)$, $k \in \mathbb{N}$. We remark that the Fourier coefficients $c(n)$ depend on $M_f(2k)$ which, for a function f defined as above, can be expressed also in terms of the a_k's and θ_k's. Therefore, a better control on these parameters will allow to estimate $M_f(2k)$ and therefore eventually to handle $\|f+1\|$ via our expression for the Fourier coefficients and Parsevall Identity.

1 Introduction

Denote as $[x]$ the "integer part of x", i.e. the greatest integer less than or equal to x and define the "fractional part" function by $\rho(x) = x - [x]$. When f is a function of the form

$$f_N(x) := \sum_{k=1}^{N} a_k\,\rho\left(\frac{\theta_k}{x}\right), \tag{1}$$

where $N \in \mathbb{N}$, $a_k \in \mathbb{C}$ and $0 < \theta_k \leqslant 1$ for all $k \in \mathbb{N}$, an elementary computation shows

$$\int_0^1 (f_N(x)+1)\,x^{s-1}\,dx = \frac{\sum_{k=1}^{N} a_k\,\theta_k}{s-1} + \frac{1}{s}\left(1 - \zeta(s)\sum_{k=1}^{N} a_k\,\theta_k^s\right); \tag{2}$$

for $\sigma > 0$, where, as usual, we denote $s = \sigma + it$. See, for instance, [1, p. 253] for a proof. We will assume that function f_N (1) satisfies also the additional condition

$$\sum_{k=1}^{N} a_k\,\theta_k = 0. \tag{3}$$

Identity (2) is the starting point of a theorem by Beurling. See [1, p. 252] for a proof and further references. As in the proof of (the easy half of) Beurling's Theorem, application of Schwarz inequality to the left side of (2) allows to show that a sufficient condition for Riemann Hypothesis is that $\|f(x)+1\|$ be done arbitrarily small for a convenient choice of a_k and θ_k, where $\|.\|$ denotes the norm in $L^2([0,1],dx)$.

Just for reference, we will call a function f_N as in (1) as an *approximation* or *Beurling function*, and the sequence $\{f_N\}_{N\in\mathbb{N}}$ is called an *approximation sequence*. We remark that approximation sequences do not necessarily *converge* to -1 in $L^2([0,1],dx)$; see the excellent work [2] on this topic. For abuse of notation and language, when f_N is a Beurling function we will call f_N+1 also a Beurling function.

A method to compute $\|f_N+1\| := \left(\int_0^1 |f_N(t)+1|^2 \, dt\right)^{1/2}$ would be to use not this definition but the Parsevall Identity $\|f_N+1\|^2 = \sum_{n\in\mathbb{Z}} |c(N,n)|^2$. In the Sec. 2 we give an expression for the Fourier coefficients $c(N,n)$ of the Beurling function f_N+1.

In this note, unless explicit statement on contrary, we assume condition (3) on a Beurling function. At a certain point we will assume also that $\theta_k = 1/b_k$ where $b_k \in \mathbb{N}$ and $|a_k| \leqslant 1$. This restriction on the θ_k's is not serious at all after Theorem 1.1 in [3]. On the other hand, the restriction on the a_k's include some of the so-called *natural approximations* considered in [2].

2 The Fourier Coefficients for a Beurling function

For convenience we define $F_N := f_N(x)+1$. We extend F_N to an *odd* function in $[-1,1]$. Therefore, $\int_{-1}^1 F_N(x)\,\cos(n\pi x)\,dx = 0$, and

$$c(N,n) := \int_{-1}^1 F_N(x)\sin(n\pi x)\,dx = 2\int_0^1 F_N(x)\sin(n\pi x)\,dx$$

$$= 2\left[\int_0^1 \sin(n\pi x)\,dx + \sum_{k=1}^N a_k \int_0^1 \rho\left(\frac{\theta_k}{x}\right)\sin(n\pi x)\,dx\right]$$

$$= 2\left[-\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \sum_{k=1}^N a_k\left(\sum_{j=1}^\infty \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \rho\left(\frac{\theta_k}{x}\right)\sin(n\pi x)\,dx + \int_{\theta_k}^1 \rho\left(\frac{\theta_k}{x}\right)\sin(n\pi x)\,dx\right)\right]$$

$$= 2\left[\frac{1}{n\pi}(1-\cos(n\pi))\right.$$
$$\left. + \sum_{k=1}^{N} a_k \left(\sum_{j=1}^{\infty} \theta_k \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \frac{\sin(n\pi x)}{x}\, dx - j \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \sin(n\pi x)\, dx + \theta_k \int_{\theta_k}^{1} \frac{\sin(n\pi x)}{x}\, dx \right)\right]$$
$$= \frac{2}{n\pi}(1-\cos(n\pi)) \; + \; 2\sum_{k=1}^{N} a_k \left(\theta_k \int_0^1 \frac{\sin(n\pi x)}{x}\, dx - \sum_{j=1}^{\infty} j \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \sin(n\pi x)\, dx \right)$$
$$= \frac{2}{n\pi}(1-\cos(n\pi))+2\left(\sum_{k=1}^{N} a_k\, \theta_k\right)\left(\int_0^1 \frac{\sin(n\pi x)}{x}\, dx\right) - 2\sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \; \frac{-\cos(n\pi x)}{n\pi}\Bigg|_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}}$$

Denote the first term in last line by $A_1 := \frac{2}{n\pi}(1-\cos(n\pi))$, and note that the second term vanishes because (3), hence

$$c(N,n) = A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \left[\cos\left(\frac{n\pi\theta_k}{j}\right) - \cos\left(\frac{n\pi\theta_k}{j+1}\right)\right]. \tag{4}$$

Now, if L is any natural number, replacing in (4) the expression for the L-th Taylor approximation for $\cos x$ given by

$$\cos x = \sum_{l=0}^{L} (-1)^l \frac{x^{2l}}{(2l)!} \; + \; \frac{1}{L!} \int_0^x \cos^{(L+1)}(t)\, (x-t)^L\, dt, \tag{5}$$

we get

$$c(N,n)$$
$$= A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \left(\sum_{l=0}^{L} (-1)^l \frac{(n\pi\theta_k)^{2l}}{(2l)!} \left[\frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}}\right] \; + \; R(L,j,n,k) \right)$$
$$= A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \left(\sum_{l=1}^{L} (-1)^l \frac{(n\pi\theta_k)^{2l}}{(2l)!} \sum_{j=1}^{\infty} j \left[\frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}}\right] \; + \; \sum_{j=1}^{\infty} j\, R(L,j,n,k) \right); \tag{6}$$

where

$$R(L,j,n,k) := \frac{1}{L!}\int_0^{\frac{n\pi\theta_k}{j}} \cos^{(L+1)}(t)\,\left(\frac{n\pi\theta_k}{j}-t\right)^L dt \\ - \frac{1}{L!}\int_0^{\frac{n\pi\theta_k}{j+1}} \cos^{(L+1)}(t)\,\left(\frac{n\pi\theta_k}{j+1}-t\right)^L dt. \quad (7)$$

Now, observe that

$$\begin{aligned}\sum_{j=1}^{\infty} j\,\left[\frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}}\right] &= \lim_{J\to\infty}\sum_{j=1}^{J}\left[\frac{j}{j^{2l}} - \frac{j+1-1}{(j+1)^{2l}}\right] \\ &= \lim_{J\to\infty}\left(\sum_{j=1}^{J}\left[\frac{j}{j^{2l}} - \frac{j+1}{(j+1)^{2l}}\right] \;+\; \sum_{j=1}^{J}\frac{1}{(j+1)^{2l}}\right) \\ &= \lim_{J\to\infty}\left(1 \;-\; \frac{J+1}{(J+1)^{2l}} \;+\; \sum_{j=2}^{J+1}\frac{1}{j^{2l}}\right) = 1 \;+\; \sum_{j=2}^{\infty}\frac{1}{j^{2l}} = \zeta(2l); \quad (8)\end{aligned}$$

Substituting (8) in (6) and denoting $R(L,n,k) := \sum_{j=1}^{\infty} j\,R(L,j,n,k)$, we get

$$\begin{aligned}c(N,n) &= A_1 + \frac{2}{n\pi}\sum_{k=1}^{N} a_k\left(\sum_{l=1}^{L}(-1)^l\,\frac{(n\pi\theta_k)^{2l}}{(2l)!}\,\zeta(2l) \;+\; R(L,n,k)\right) \\ &= A_1 + \frac{2}{n\pi}\sum_{l=1}^{L}(-1)^l\,\frac{(n\pi)^{2l}}{(2l)!}\,\zeta(2l)\left(\sum_{k=1}^{N} a_k\,\theta_k^{2l}\right) \;+\; \sum_{k=1}^{N} a_k\,R(L,n,k). \quad (9)\end{aligned}$$

If we denote $M_g(s) := \int_0^1 g(x)\,x^{s-1}\,dx$, then by (2), under condition (3), we have

$$\sum_{k=1}^{N} a_k\,\theta_k^{2l} = \frac{1}{\zeta(2l)}\left(1 \;-\; (2l)\,M_{F_N}(2l)\right). \quad (10)$$

Substituting now (10) in (9) we get

$$
\begin{aligned}
c(N,n) = A_1 \;+\; & \frac{2}{n\pi}\sum_{l=1}^{L}(-1)^l\,\frac{(n\pi)^{2l}}{(2l)!} \\
& -\; \frac{2}{n\pi}\sum_{l=1}^{L}(-1)^l\,\frac{(n\pi)^{2l}}{(2l)!}\,(2l)\,M_{F_N}(2l) + \sum_{k=1}^{N} a_k\,R(L,n,k) \\
& = A_1 \;+\; \frac{2}{n\pi}\sum_{l=1}^{L}(-1)^l\,\frac{(n\pi)^{2l}}{(2l)!} \\
& \qquad +\; 2\sum_{l=1}^{L}(-1)^{l-1}\,\frac{(n\pi)^{2l-1}}{(2l-1)!}\,M_{F_N}(2l) + \sum_{k=1}^{N} a_k\,R(L,n,k). \qquad (11)
\end{aligned}
$$

This is an exact expression valid for all $L \in \mathbb{N}$ but a better expression arises in the limit $L \to \infty$. In this case, the first term cancels the second and is not difficult to prove the following

Lemma 1 *For each $k \in \mathbb{N}$ assume $|a_k| \leqslant 1$ and $\theta_k = 1/b_k$ with $b_k \in \mathbb{N}$. Then, for any fixed N and n in $\mathbb{N}$ is $\lim_{L\to\infty} \sum_{k=1}^{N} a_k\,R(L,n,k) = 0$.*

(The proof will be given in Sec. 4). Therefore, the final expression for the Fourier coefficient is

$$
c(N,n) = 2\sum_{l=1}^{\infty}(-1)^{l-1}\,\frac{(n\pi)^{2l-1}}{(2l-1)!}\,M_{F_N}(2l). \qquad (12)
$$

3 An Expression for the Mellin Transform

Directly from (12) we have

$$
\begin{aligned}
M_{F_N}(s) = \int_0^1 F_N(x)\,x^{s-1}\,dx &= \int_0^1 \left(\sum_{n=1}^{\infty} c(N,n)\sin(n\pi x)\right)\,x^{s-1}\,dx \\
&= \sum_{n=1}^{\infty}\left(\int_0^1 \sin(n\pi x)\,x^{s-1}\,dx\right)\,2\sum_{l=1}^{\infty}(-1)^{l-1}\,\frac{(n\pi)^{2l-1}}{(2l-1)!}\,M_{F_N}(2l). \qquad (13)
\end{aligned}
$$

The last expression gives the value of $M_{F_N}(s)$ in term of its values in the even integers $M_{F_N}(2l)$.

Note also that under the hypothesis of Lemma 1 we have $\left|\sum_{k=1}^N a_k\, \theta_k^{2l}\right| \leqslant \zeta(2l)$, and this combined with (10), or (2), gives an estimation on the Mellin transform

$$|M_{F_N}(2l)| \leqslant \frac{1+\zeta^2(2l)}{2l}. \tag{14}$$

As in the derivation of relation (18) in Sec. 4, a better control on the a_k's and θ_k's will allows to control $M_{F_N}(2l)$ via (10), and therefore to control $\|f_N(x)+1\|$ via (12) and Parsevall Identity.

4 Proof of Lemma 1

For sake of brevity, denote $I(j) := (1/L!) \int_0^{\frac{n\pi\theta_k}{j}} \cos^{(L+1)}(t)\, (\frac{n\pi\theta_k}{j} - t)^L$. From (7) we have

$$\begin{aligned} R(L,n,k) &:= \sum_{j=1}^{\infty} j\, R(L,j,n,k) = \lim_{J\to\infty} \sum_{j=1}^{J} j\, I(j) \;-\; (j+1-1)\, I(j+1) \\ &= \lim_{J\to\infty} \left(\sum_{j=1}^{J} [j\, I(j) \;-\; (j+1)\, I(j+1)] \;+\; \sum_{j=1}^{J} I(j+1) \right) \\ &= \lim_{J\to\infty} \left(I(1) \;-\; (J+1)\, I(J+1) \;+\; \sum_{j=1}^{J} I(j+1) \right) \\ &= \lim_{J\to\infty} \left(\sum_{j=1}^{J+1} I(j) \;-\; (J+1)\, I(J+1) \right). \end{aligned} \tag{15}$$

Using the elementary estimative

$$|I(j)| \leqslant \frac{1}{L!}\, \left. \frac{-\left(\frac{n\pi\theta_k}{j} - t\right)^{L+1}}{L+1} \right|_0^{\frac{n\pi\theta_k}{j}} = \frac{1}{j^{L+1}}\, \frac{(n\pi\theta_k)^{L+1}}{(L+1)!}, \tag{16}$$

for any $L \in \mathbb{N}$ we have

$$|R(L,n,k)| \;\leqslant\; \lim_{J\to\infty} \frac{(n\pi\theta_k)^{L+1}}{(L+1)!} \left(\sum_{j=1}^{J+1} \frac{1}{j^{L+1}} \;+\; \frac{1}{(J+1)^L} \right) \;=\; \frac{(n\pi\theta_k)^{L+1}}{(L+1)!}\, \zeta(L+1). \tag{17}$$

Therefore, for any $N \in \mathbb{N}$,

$$\left|\sum_{k=1}^{N} a_k\, R(L,n,k)\right| \leqslant \sum_{k=1}^{N} |R(L,n,k)| \leqslant \frac{(n\pi)^{L+1}}{(L+1)!}\,\zeta(L+1)\sum_{k=1}^{N}\theta_k^{L+1}$$
$$= \frac{(n\pi)^{L+1}}{(L+1)!}\,\zeta(L+1)\sum_{k=1}^{N}\frac{1}{b_k^{L+1}} \leqslant \frac{(n\pi)^{L+1}}{(L+1)!}\,\zeta^2(L+1). \quad (18)$$

Now the assertion of Lemma 1 follows observing that $\zeta(L+1)$ remains close to 1 for large L and the first factor in the right side of (18) goes to zero because is the $(L+1)$-th term of the (convergent) series for $\exp(n\pi)$.

References

[1] Donoghue W. F. *Distributions and Fourier Transforms*, Academic Press, New York, 1969.

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[3] L. Bez-Duarte, *A Strengthening of the Nyman-Beurling Criterion for the Riemann Hypothesis*. arXiv:math.NT/0202141v2

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